to be reproduced in difference form. This function is substituted into (2.3), which in the long run results in an inequality of the type (1.7). Therefore, the presence of square-summable second derivatives along the tangent directions and mixed derivatives, is established. From the conditions for the equilibrium equations to be valid near the boundary we also obtain the existence of square-summable second derivatives along the normal.

Construction of the measure and its properties. We will formulate a theorem on the existence of a measure characterizing the action of one body on another. Exactly as in the problem of the interaction between an elastic and a rigid body, the case of the distinct location of $\Gamma_c, \Gamma_\sigma, \Gamma_\omega, \Gamma_{\sigma'}, \Gamma_{\omega'}$, must be examined separately.

First, for each point $x_0 \in \partial\Gamma_c$ let a neighbourhood $d(x_0)$ exist that possesses the property that $d(x_0) \cap \Gamma \subset \Gamma_c \cup \Gamma_\sigma$ and $d(x_0) \cap \Gamma' \subset \Gamma_c \cup \Gamma_{\sigma'}$. Hence, the following theorem holds:

Theorem 5. A measure μ can be defined on a σ-algebra of Borel subsets Γ_c such that for arbitrary functions $\varphi = (\gamma, \gamma') \in H \cap C(\Gamma_c)$ the following representation holds ($\psi \in K$ is the solution of (2.3)):

$$(dE(\psi), \varphi) = -\int_{\Gamma_c} (\gamma n - \gamma' n)\, d\mu \qquad (2.4)$$

The properties of the measure constructed in such a manner are determined by the smoothness of the solution. In particular, the presence of second derivatives for the solution near the contact boundary enables us to prove that the singular component of the measure μ equals zero at the points $\Gamma_c \setminus \partial\Gamma_c$. The reasoning is similar to that used at the end of Sect.1. The density of the measure μ turns out to equal $-\sigma_{ij}(\omega)\, n_j n_i$.

In conclusion, we consider the situation when a neighbourhood $d(x_0)$ exists for an arbitrary point $x_0 \in \partial\Gamma_c$ for which $d(x_0) \cap \Gamma \subset \Gamma_c \cup \Gamma_\omega$, $d(x_0) \cap \Gamma' \subset \Gamma_c \cup \Gamma_{\omega'}$.

Theorem 6. A measure μ can be defined on a σ-algebra of Borel subsets $\Gamma_c \setminus \partial\Gamma_c$ such that for any function $\varphi = (\gamma, \gamma') \in H \cap C_0(\Gamma_c)$ the representation (2.4) holds. The singular component of this measure is zero, and the density equals $-\sigma_{ij}(\omega)\, n_j n_i$, where $\mu(B) < +\infty$ for any compact $B \subset \Gamma_c \setminus \partial\Gamma_c$.

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ON THE FORMULATION AND INVESTIGATION OF A SPATIAL CONTACT PROBLEM FOR ELASTIC BODIES UNDER MIXED FRICTION CONDITIONS*

I.I. KUDISH

A spatial contact problem is formulated and investgated for rough elastic bodies which touch each other under mixed friction conditions: the elastic bodies are separated in one part of the contact domain by a layer of viscous incompressible liquid (lubricant), while in the other they are in direct contact (such conditions are characteristic for roller bearings, gear transmissions, etc.). The problem is reduced to a system of non-linear integro-differential and integral equations and inequalities in the contact domain, part of the external boundary, and a number of inner boundaries that are unknown in advance, but separate the lubricated and unlubricated zones. Special cases are problems of dry and completely

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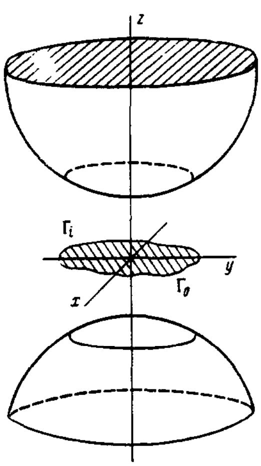
lubricated contact. A formulation is given for the problem for the case when the materials of the bodies are identical. The problem of mixed friction is considered in strongly drawn out contact. Sections of the contact domain in which the interaction between the bodies is direct or by means of the lubrication layer are investigated using asymptotic methods.

1. Formulation of the problem. We introduce a moving system of coordinates (see the sketch) in the contact domain. We direct the z-axis from the lower to the upper body so that it passes through the centres of curvature of the bodies making contact. We superpose the xy plane on the middle plane in the lubrication layer $z = 0$. Here the equations of the surfaces bounding the bodies in contact have the form $z = \pm {}^1/_2 h(x, y)$, respectively, where $h = h(x, y)$ is a function of the gap between the bodies making contact.

We will assume the contact to be local and replace the bodies making contact by half-spaces. We will approximate the micro-roughness covering the surfaces of the bodies making contact by a power-law function of the pressure p with exponent $\alpha, 0 < \alpha \leqslant 1$ /1/. It is assumed that slow stationary motion of the surfaces of the bodies making contact occurs with linear velocities $\mathbf{u}_1 = (u_1, v_1)$ and $\mathbf{u}_2 = (u_2, v_2)$. It is assumed that the slippage velocity is small compared with the rolling velocity, and that the inertial forces can be neglected compared with the viscous forces in the lubricant /2, 3/. It is also assumed that the lubricant between the bodies possesses the properties of an incompressible Newtonian liquid and is under isothermal conditions, where the layer thickness is small compared with the characteristic dimensions of the contact region /2, 3/.

With these assumptions the tangential stress vector in the lubricant layer is proportional to the gradient of the linear velocity of the lubricant particles, i.e.,

$$\boldsymbol{\tau} = \mu\, \partial \mathbf{u}/\partial z \quad (\boldsymbol{\tau} = (\tau_{xz}, \tau_{yz}),\ \mathbf{u} = (u, v)) \tag{1.1}$$



where μ is the viscosity of the lubricant.

Where there is direct contact between the elastic bodies, dry friction forces occur. In general, the direct contact domain is divided into adhesion and slippage zones in which the relative slippage of the bodies $\mathbf{s}(x, y)$ is, respectively, zero and different from zero, where the friction stress in the latter case obeys Coulomb's law

$$\boldsymbol{\tau} = fp\mathbf{s}/\,|\mathbf{s}|,\quad |\mathbf{s}| > 0 \tag{1.2}$$

($f = f(p, \ |\mathbf{s}|)$ is the coefficient of friction. The inequality

$$|\boldsymbol{\tau}| \leqslant fp,\quad |\mathbf{s}| = 0 \tag{1.3}$$

is here satisfied in the adhesion zone.

We will examine the boundary conditions for the liquid velocity $\mathbf{u}$. Because of the non-penetration and adhesion conditions, and taking into account the assumption that $\operatorname{grad} h$ is small, we obtain for the components of the particle velocity $\mathbf{w} = (u, v, w)$ of the liquid on the friction surfaces

$$\begin{aligned} &\mathbf{u} = \mathbf{u}_1,\ w = -{}^1/_2\,(\mathbf{u}_1, \operatorname{grad} h),\ z = -h/2 \\ &\mathbf{u} = \mathbf{u}_2,\ w = {}^1/_2\,(\mathbf{u}_2, \operatorname{grad} h),\ z = h/2 \end{aligned} \tag{1.4}$$

Carrying out standard computations /3/ taking the above assumptions and conditions (1.4) into account, we find after integrating the continuity equation $\operatorname{div} \mathbf{w} = 0$ with respect to z between the limits $-h/2$ and $h/2$

$$\frac{\partial}{\partial x} Q_x + \frac{\partial}{\partial y} Q_y = 0,\quad \mathbf{Q} = (Q_x, Q_y) = \int_{-h/2}^{h/2} \mathbf{u}(x, y, z)\, dz \tag{1.5}$$

Furthermore, integrating the equations of motion taking into account a number of the assumptions made above and conditions (1.4), we find

$$\mathbf{u} = \frac{\mathbf{u}_1 + \mathbf{u}_2}{2} + \frac{\mathbf{u}_2 - \mathbf{u}_1}{h} z + \frac{1}{2\mu}\left(z^2 - \frac{h^2}{4}\right) \operatorname{grad} p,\quad p = p(x, y) \tag{1.6}$$

and we obtain for the liquid mass flow rate

$$\mathbf{Q} = \frac{\mathbf{u}_1 + \mathbf{u}_2}{2} h - \frac{h^3}{12\mu} \operatorname{grad} p \tag{1.7}$$

Therefore, the Reynolds equation takes the form

$$\frac{\partial}{\partial x}\left(\frac{h^3}{12\mu}\frac{\partial p}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{h^3}{12\mu}\frac{\partial p}{\partial y}\right)=\frac{u_1+u_2}{2}\frac{\partial h}{\partial x}+\frac{v_1+v_2}{2}\frac{\partial h}{\partial y} \tag{1.8}$$

The smallness of the slippage velocity compared with the rolling velocity was also taken into account in deriving (1.8). Equation (1.8) obviously holds in those zones of the contact domain in which $h > 0$.

For simplicity, we will assume, when deriving the equation to determine the gap h and when formulating the problem of determining the tangential stress τ and the slippage velocity s, that the materials of the elastic bodies are identical. The problem of determining the contact pressure and the gap then becomes separate from the problem of determining the tangential stress and the slippage velocity in the contact domain.

We will write down the difference in the elastic displacements $W = (U_2 - U_1, V_2 - V_1, W_2 - W_1)$ of points on the upper and lower body surfaces. Following /4/, we obtain

$$U_2 - U_1 = \frac{1}{2\pi G}\iint\limits_E \left[\frac{1-\nu\sin^2\theta}{R}(\tau_{xz}^+ - \tau_{xz}^-) + \frac{\nu\sin\theta\cos\theta}{R}(\tau_{yz}^+ - \tau_{yz}^-)\right]dx'\,dy' \tag{1.9}$$

$$V_2 - V_1 = \frac{1}{2\pi G}\iint\limits_E \left[\frac{\nu\sin\theta\cos\theta}{R}(\tau_{xz}^+ - \tau_{xz}^-) + \frac{1-\nu\cos^2\theta}{R}(\tau_{yz}^+ - \tau_{yz}^-)\right]dx'\,dy'$$

$$W_2 - W_1 = kp^\alpha + \frac{1-2\nu}{4\pi G}\iint\limits_E \left[\frac{\cos\theta}{R}(\tau_{xz}^+ + \tau_{xz}^-) + \frac{\sin\theta}{R}(\tau_{yz}^+ + \tau_{yz}^-)\right]dx'\,dy' + \frac{1-\nu}{\pi G}\iint\limits_E \frac{p\,dx'\,dy'}{R}$$

Here $\tau^+ = (\tau_{xz}^+, \tau_{yz}^+)$ and $\tau^- = (\tau_{xz}^-, \tau_{yz}^-)$ are the tangential stresses acting on the upper and lower surfaces making contact, respectively, and determined by the relationships (1.1)–(1.3). The contact domain $E = E_p \cup E_\tau$, where E_p is a zone of the contact domain in which $p > 0$ and E_τ is a zone of the contact domain in which $|\tau| > 0$.

From the kinematic relationships we obtain* for the slippage velocity s, neglecting second-order infinitesimals in $|u_2 - u_1|$,

$$\mathbf{s} = {}^1/_2(\mathbf{u}_1 + \mathbf{u}_2, \nabla)(\mathbf{U}_2 - \mathbf{U}_1) + \mathbf{u}_2 - \mathbf{u}_1 \tag{1.10}$$

$$\mathbf{U}_2 = (U_2, V_2),\quad \mathbf{U}_1 = (U_1, V_1),\quad \nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$$

Now taking account of the collapse of the microroughness, and we find for the gap h (see (1.1), (1.6) and (1.9))

$$h = h_0 + kp^\alpha + \frac{x^2}{R_x'} + \frac{y^2}{R_y'} + \frac{1-\nu}{\pi G}\iint\limits_{E_p}\frac{p\,dx'\,dy'}{R} - \frac{1-2\nu}{4\pi G}\iint\limits_{E_p}\left[\cos\theta\frac{\partial p}{\partial x'} + \sin\theta\frac{\partial p}{\partial y'}\right]\frac{h\,dx'\,dy'}{R} \tag{1.11}$$

$$\sin\theta = \frac{y-y'}{R},\quad \cos\theta = \frac{x-x'}{R},\quad R = \sqrt{(x-x')^2 + (y-y')^2}$$

Here h_0 is a constant not known in advance, k and α are coefficients in the law of collapse of the microroughness, R_x' and R_y' are the reduced radii of curvature of the bodies making contact, and G and ν are the shear modulus and Poisson's ratio for the body materials.

In the zone where there is no lubricant, we have $h = 0$, where h is determined from (1.11).

The statics condition and the boundary conditions

$$\iint\limits_{E_p} p(x', y')\,dx'\,dy' = P;\quad p|_\Gamma = 0 \tag{1.12}$$

must be supplemented by the equations and inequalities mentioned, and Γ_i is the given boundary of the entrance into the contact domain E_p if

$$h|_{\Gamma_i} > 0 \quad \text{and} \quad (\mathbf{Q}, \mathbf{n})|_{\Gamma_i} < 0$$

$$\frac{dp}{d\mathbf{n}}\bigg|_{\Gamma_\bullet} = 0,\quad \text{if} \quad h|_{\Gamma_\bullet} > 0 \quad \text{and} \quad (\mathbf{Q}, \mathbf{n})|_{\Gamma_\bullet} \geqslant 0$$

* Gol'dshtein R.V., Zazovskii A.F., Spektor A.A. and Fedorenko R.P., Solution of spatial contact problems of rolling with slippage and adhesion by a variational method. Preprint No.134, Inst. Problem Mekhaniki Akad. Nauk SSSR, Moscow, 66p., 1979.

Here P is the compressive force of the bodies making contact, Γ is the boundary of the contact domain, E_p, Γ_i is the boundary of the entrance domain, a part of the boundary Γ, Γ_0 is the boundary of the exit domain, a part of the boundary Γ, and $\mathbf{n}$ is the external unit normal to the boundary Γ.

Therefore, to determine p and h we obtain the following relationships: if $h > 0$, then (1.8) and (1.11) hold, otherwise $h = 0$ and (1.11) is used to determine p for $h = 0$; here the function h is calculated using (1.11). The conditions (1.12) are added to the relations mentioned.

After having determined the pressure p and the gap h in the case of identical or incompressible materials of the bodies, it is then possible to find the friction stress and the slippage velocity. By using relations (1.1)–(1.3), (1.6), (1.9) and (1.10), the problem of determining the tangential stress and the slippage velocity in the contact domain reduces to a system of non-linear equations and inequalities

$$\begin{aligned}
&\tau = \mu \mathbf{s}/h \text{ when } h > 0 \\
&|\tau| \leqslant fp \quad (|\mathbf{s}| = 0), \quad \tau = fp\mathbf{s}/|\mathbf{s}| \quad (|\mathbf{s}| > 0) \text{ when } h = 0 \\
&\mathbf{s} = -B(\tau) + \mathbf{v}, \quad |\tau|\,|_{\partial E_\tau} = 0 \\
&B(\tau) = \frac{u_1 + u_2}{2} \iint\limits_{E_\tau} D^1(x - x', y - y')\,\tau(x', y')\,dx'\,dy' + \\
&\frac{v_1 + v_2}{2} \iint\limits_{E_\tau} D^2(x - x', y - y')\,\tau(x', y')\,dx'\,dy'
\end{aligned} \tag{1.13}$$

The elements of the matrices D^1 and D^2 have the form *

$$\begin{aligned}
&D_{11}^1 = -\frac{\cos\theta\,(3\nu\sin^2\theta - 1)}{\pi G R^2}, \quad D_{12}^1 = D_{21}^1 = -\frac{\nu\sin\theta\,(1 - 3\cos^2\theta)}{\pi G R^2} \\
&D_{22}^1 = -\frac{\cos\theta\,(\nu - 1 - 3\nu\sin^2\theta)}{\pi G R^2}, \quad D_{11}^2 = -\frac{\sin\theta\,(\nu - 1 - 3\nu\cos^2\theta)}{\pi G R^2} \\
&D_{12}^2 = D_{21}^2 = -\frac{\nu\cos\theta\,(1 - 3\sin^2\theta)}{\pi G R^2} \\
&D_{22}^2 = -\frac{\sin\theta\,(3\nu\cos^2\theta - 1)}{\pi G R^2}
\end{aligned} \tag{1.14}$$

($\mathbf{v} = \mathbf{u}_2 - \mathbf{u}_1$ is the velocity vector of "rigid" slippage, and ∂E_τ is the boundary of the domain E_τ not known in advance),

After determining the slippage vector in the zone where there is lubricant, i.e., $h > 0$, the friction stress on the surfaces can be represented in the form

$$\tau^\pm = \mp \frac{\mu \mathbf{s}}{h} - \frac{h}{2}\,\mathrm{grad}\,p \tag{1.15}$$

We will make a number of remarks of a physical nature concerning the laws of friction (1.1)–(1.3). Because of adsorption effects, the boundary layers of lubricants acquire the properties of structural anisotropic fluids. As a number of experimental investigations /5/ shows, a continuous transition from liquid (relationship (1.1)) to dry friction (relationships (1.2) and (1.3)) occurs in a small number of molecular liquid layers on the boundary of the solid. The specific features of this transition have been studied to only a small extent and depend on the adsorption properties of the lubricant-solid pair.

We will later require continuity of the passage of (1.1) into (1.2) and (1.3), i.e. continuity of the tangential stresses τ on the lines separating the zones with $h > 0$ and $h = 0$.

We will obtain certain corollaries resulting from the condition of continuity of the tangential stresses. We consider the part of the boundary between the lubricated and unlubricated contact zones on which slippage occurs, i.e., $|\mathbf{s}| > 0$. Then using (1.2) and (1.5) and starting from the continuity of the tangential stress, we obtain $\mu = \mu(p, h)$ and

$$\lim_{h \to 0} \mu/h = fp/|\mathbf{s}|, \quad |\mathbf{s}| > 0 \tag{1.16}$$

On the other hand, for an analogous section of the boundary $h = 0$ on which adhesion $|\mathbf{s}| = 0$ holds, by taking account of the continuity of the friction stress and (1.3) and (1.15) we find

$$\lim_{h \to 0,\, |\mathbf{s}| \to 0} (\mu|\mathbf{s}|/h - fp) \leqslant 0, \quad |\mathbf{s}| = 0 \tag{1.17}$$

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* See the previous footnote.

Relations (1.16) and (1.17) indicate the structural properties of the lubricant boundary layers; the former is confirmed by the graph of the dependence of the coefficient of friction on the layer thickness (/5/, p.316).

Therefore, the relations $\mu = \mu(p, h)$ and $f = f(p, |\mathbf{s}|)$ that satisfy (1.16) and (1.17) on the boundaries of the "lubricated" and "unlubricated" zones, must be appended to the formulation described above for the problem of determining τ and $\mathbf{s}$ (1.13), (1.14) taking the continuity of τ into account on the boundaries of zones with $h > 0$ and $h = 0$.

2. The case of contact strongly drawn out in the direction of the y axis. We will introduce the dimensionless variables

$$x' = \frac{x}{a_H}, \quad y' = \frac{y}{b_H}, \quad p' = \frac{p}{p_H}, \quad h' = \frac{h}{h_0}, \quad \mu' = \frac{\mu}{\mu_0}$$

$$\tau' = \frac{\tau}{f_0 p_H}$$

$$\mathbf{s}' = \frac{2\mathbf{s}}{|\mathbf{u}_1 + \mathbf{u}_2|}, \quad \mathbf{v}' = \frac{2\mathbf{v}}{|\mathbf{u}_1 + \mathbf{u}_2|}, \quad \mathbf{Q}' = \frac{2R_x'\mathbf{Q}}{|\mathbf{u}_1 + \mathbf{u}_2| a_H^2}$$

$$\varphi(\delta) = \frac{K(e) - D}{\delta^4 D}, \quad D = \frac{1}{e^2}[K(e) - E(e)], \quad e = \sqrt{1 - \delta^2}$$

$$\theta_0 = \frac{1 - 2\nu}{4(1 - \nu)} \frac{a_H}{R_x'}, \quad V = \frac{6\mu_0 |\mathbf{u}_1 + \mathbf{u}_2| R_x'^2}{p_H a_H^3}, \quad \lambda = \frac{k p_H^{\alpha} R_x'}{a_H^2}$$

$$\delta = \frac{a_H}{b_H}, \quad H_0 = \frac{h_0 R_x'}{a_H^2}, \quad \operatorname{tg} \gamma = \frac{v_1 + v_2}{u_1 + u_2}$$

$$\psi_h = \frac{1}{12(1 - \nu)\pi\delta^3 D}\left(\frac{a_H}{R_x'}\right)^2, \quad \psi_0 = \frac{1}{(1 - \nu)\pi\delta^3 D} \frac{a_H}{R_x'}$$

Here $K(e)$ and $E(e)$ are the complete elliptic integrals of the first and second kinds, the constant δ is determined from the equation $\delta^2\varphi(\delta) = R_x'/R_y'$, a_H and b_H are the semi-axes of the Hertz contact ellipse, and p_H is the maximum Hertz stress.

We will consider the case of strongly drawn out contact when $\delta \ll 1$, $\gamma \sim \delta$, and $v_y \sim \delta$. It is more convenient to write the equation for h in a form in which H_0 is the dimensionless thickness of the layer at the exit from the contact domain, i.e., at the point $(x_\Gamma, y) \in \Gamma_0$. Here, we have $H_0 = H_0(y)$. When $\delta \ll 1$ the integrals in the equations of the problem can be simplified as in /6/. Then taking account of the above-mentioned transformation of the equation for h we write the problem of determining the principal terms in the asymptotics of p and h in the form

$$\frac{\partial}{\partial x}\left(\frac{h^3}{\mu}\frac{\partial p}{\partial x}\right) = \frac{V}{H_0^2}\frac{\partial h}{\partial x}, \quad H_0 > 0 \tag{2.1}$$

$$\lambda p^{\alpha} + x^2 - c_p^2 + \frac{2}{\pi\delta^3 D}\int_{a_p}^{c_p} p(t, y) \ln\frac{c_p - t}{|x - t|}\, dt = 0, \quad H_0 = 0 \tag{2.2}$$

$$\int_{a_p}^{c_p} p(t, y)\, dt = \frac{\pi}{2} p_{00}(y), \quad p(a_p, y) = p(c_p, y) = 0 \tag{2.3}$$

$$\partial p(c_p, y)/\partial x = 0, \quad H_0 > 0 \tag{2.4}$$

$$H_0(h - 1) = \lambda p^{\alpha} + x^2 - c_p^2 + \frac{2}{\pi\delta^3 D}\int_{a_p}^{c_p} p(t, y)\ln\frac{c_p - t}{|x - t|}\, dt + \frac{H_0\theta_0}{\delta^3 D}\int_{x}^{c_p} h\frac{\partial p(t, y)}{\partial t}\, dt \tag{2.5}$$

It is taken into account in (2.1)–(2.5) that if the gap h vanishes at a certain point, then within the framework of the plane problem this will imply $h = 0$ in the whole contact domain, and therefore, $H_0 = 0$. This results from the fact that the liquid mass flow rate through the gap h into the whole domain is constant in the plane case. Moreover, the following notation is introduced in (2.1)–(2.5): a_p and c_p are, respectively, the abscissas of the entrance and exit points of the contact domain in the section $y = \text{const}$ in which $p > 0$, i.e., $a_p = a_p(y)$ and $c_p = c_p(y)$, and $p_{00}(y)$ is a function unknown in advance the characterizes the force applied to this section.

In exactly the same way, we obtain $\tau_y = s_y = 0$ for the principal terms of the asymptotics in the problem of determining τ and $\mathbf{s}$ for $\delta \ll 1$, $\gamma \sim \delta$, and $v_y \sim \delta$ and

$$\tau_x = \mu s_x/h \text{ when } H_0 > 0 \tag{2.6}$$

$$|\tau_x| \leqslant fp \ (s_x = 0), \quad \tau_x = fp \operatorname{sign} s_x \ (|s_x| > 0) \text{ when } H_0 = 0$$

$$s_x = -2\psi_0(1-\nu)\int_{a_\tau}^{c_\tau} \frac{\tau_x(t, y)}{t-x}\,dt + v_*, \quad \tau_x(a_\tau, y) = \tau_x(c_\tau, y) = 0$$

Here a_τ and c_τ are, respectively, the coordinates of the beginning and end of the contact domain outside which we have $\tau_x = 0$.

The systems of equations and inequalities (2.1)–(2.5) and (2.6) hold everywhere in the contact domain outside small neighbourhoods, of the order of δ, of the point $y = \pm 1$ and the other contact zones abutting on the boundaries with radius of curvature of the order of δ. Moreover, it follows from the systems mentioned that in general the contact domain is divided into a number of alternating strips filled with lubricant and strips in which there is no lubricant.

Starting from (2.1)–(2.6) and relations (1.16), (1.17), the continuity of the pressure p and the slippage velocity vector component s_x can be shown for the passage through the section $y = \text{const}$ in which H_0 vanishes.

The problem of determining p and h will be investigated below.

We will consider the case when there is no lubricant anywhere in the contact domain i.e., $H_0 = 0$. It is here necessary to solve (2.2) and (2.3). Taking into account that the shape of the bodies making contact is described by an even function, we obtain $c_p = -a_p$. Then by transformation of variables

$$(x, -c_p, c_p) = \sigma(x_0, -c_0, c_0),$$
$$p = p_{00}(y)\,p^\circ, \quad \lambda = \sigma^2 p_{00}^{-\alpha}(y)\,\lambda_0, \quad \sigma = \frac{p_{00}(y)}{\delta^3 D} \tag{2.7}$$

we reduce (2.2) and (2.3) to a form agreeing, apart from the notation, with the form of the corresponding equations of the problem with a previously unknown contact domain in /7/. Consequently, the results presented in /7/ can be obtained when studying (2.2) and (2.3) by the methods of merging and regular asymptotic expansions. In particular, when $\lambda_0 = \lambda_0(\delta) \ll 1, \alpha > 3$ and $\lambda_0 = \lambda_0(\delta) \gg 1$ the solution of the problem is constructed by regular asymptotic methods in analytic form, and when $\lambda_0 = \lambda_0(\delta) \ll 1$ and $0 < \alpha \leqslant 3$ by the method of merging asymptotic expansions.

We will consider the case when there is a non-zero lubricant layer everywhere in the contact domain, i.e., $H_0 > 0$. It is here necessary to investigate (2.1), (2.3)–(2.5). By transformation of variables (2.7)

$$H_0 = \sigma^2 H_{00}, \quad V = p_{00}(y)\,\sigma^3 V_0, \quad \theta_0 = \frac{1}{\sigma}\,\theta_{00} \tag{2.8}$$

(2.1), (2.3)–(2.5) can be reduced to equations analogous (taking the notation into account) to the equations in /8/ for $F(x) = x$. The asymptotic analysis of the equations mentioned is described in /8/ for the case of heavy loading, i.e., when $V_0 = V_0(\delta) \ll 1$ or $\mu(p^\circ, h) \gg 1$ for $p^\circ \sim 1$.

Omitting the subscripts 0 and p, we will write the equations of the problem that have been transformed taking (2.7) and (2.8) into account, in the form

$$\frac{\partial}{\partial x}\left(\frac{h^3}{\mu}\frac{\partial p}{\partial x}\right) = \frac{V}{H_0^3}\frac{\partial h}{\partial x} \tag{2.9}$$

$$H_0(h-1) = \lambda p^\alpha + x^2 - c^2 + \frac{2}{\pi}\int_a^c p(t, y)\ln\frac{c-t}{|x-t|}\,dt + H_0\theta_0\int_x^c h\,\frac{\partial p(t, y)}{\partial t}\,dt \tag{2.10}$$

$$p(a, y) = p(c, y) = \frac{\partial p(c, y)}{\partial x} = 0, \quad \int_x^c p(t, y)\,dt = \frac{\pi}{2} \tag{2.11}$$

It follows from (2.10) that $h(c, y) = 1$. Consequently, by differentiating (2.10) with respect to x, we obtain a differential equation in $h(x, y)$. Then by integrating it with respect to h, taking the condition $h(c, y) = 1$ into account, we find

$$H_0(h-1) = \lambda p^\alpha - H_0\theta_0 p + x^2 - c^2 + \frac{2}{\pi}\int_a^c p(t, y)\ln\frac{c-t}{|x-t|}\,dt -$$
$$\theta_0 e^{-\theta_0 p}\int_c^x \frac{\partial p(t, y)}{\partial t}\,e^{\theta_0 p}\left[\lambda p^\alpha - H_0\theta_0 p + t^2 - c^2 + \frac{2}{\pi}\int_a^c p(s, y)\ln\frac{c-s}{|t-s|}\,ds\right]dt \tag{2.12}$$

We will consider (2.9), (2.12), (2.11) in the case of heavy loading. We shall regard the contact as being under heavy loading if the small parameter $\omega = \omega(\delta)$ enters the equations of the problem (its physical meaning is indicated above), and the estimate /8/

$$H_0(h-1) \ll 1 \quad \text{when} \quad x-a \gg \varepsilon_q,\ c-x \gg \varepsilon_g \tag{2.13}$$

holds in the domain far from the contact boundaries.

Here $\varepsilon_q = \varepsilon_q(\omega) \ll 1$ and $\varepsilon_g = \varepsilon_g(\omega) \ll 1$ are the characteristic dimensions of the entrance and exit zones of the contact domains that are boundary layers in the neighbourhoods of the points $x = a$ and $x = c$.

We determine the asymptotic form of the solution of the problem in the outer domain, i.e., far from the contact boundary. Then, by integrating (2.10) with respect to h taking the estimate (2.13) into account in the outer domain, we obtain equations for the degenerate solution

$$z - \theta_0 \int_c^x \frac{\partial p_0(t,y)}{\partial t} z(t,y)\,dt = \lambda p_0^\alpha(c,y)\, e^{\theta_0 p_0(c,y)} \tag{2.14}$$

$$z(x,y) = e^{\theta_0 p_0}\Big[\lambda p_0^\alpha - H_0\theta_0 p_0 + x^2 - c^2 +$$

$$\frac{2}{\pi}\int_a^c p_0(t,y)\ln\frac{c-t}{|x-t|}\,dt + H_0\theta_0 p_0(c,y)\Big], \quad \int_a^c p_0(t,y)\,dt = \frac{\pi}{2}$$

which can be reduced by differentiation and subsequent transformation to the form

$$\lambda p_0^\alpha - H_0\theta_0 p_0 + x^2 - c^2 + \frac{2}{\pi}\int_a^c p_0(t,y)\ln\frac{c-t}{|x-t|}\,dt = \tag{2.15}$$

$$\lambda p_0^\alpha(c,y) - H_0\theta_0 p_0(c,y)$$

$$\int_a^c p_0(t,y)\,dt = \frac{\pi}{2}$$

We will consider the case when $\theta_0 = 0$ and $0 < \alpha < 1$. A further asymptotic analysis evidently agrees completely with that given in /8/ in which two closed systems of equations for the principal terms of the pressure and gap asymptotics are obtained that hold asymptotically in the entrance and exit zones. Moreover, an estimate is obtained in /8/ for the thickness of the lubricant layer H_0

$$H_0 = A(V\varepsilon_q^2)^{1/3},\ A = A(\alpha_1, \lambda_0, \alpha) \sim 1 \quad \text{when} \quad \omega \ll 1 \tag{2.16}$$

where $\varepsilon_q \ll V^{0.4}$ for oil deficiency conditions and $\varepsilon_q = V^{0.4}$ for flood lubrication conditions.

The case of incompressible materials $(\theta_0 = 0)$ was considered above. The case of compressible materials $(\theta_0 > 0)$ can also be considered by a method differing substantially from that described in /8/ for $\alpha = 1$ and $H_0\theta_0 = \lambda$.

It should be noted that the function $p_{00}(y)$ in the statics condition (2.3) and relations (2.7) and (2.8) can be determined by an asymptotic method irrespective of whether the bodies are smooth or rough.

Light loading conditions, for which direct contact does not occur, as a rule, and $V \gg 1$ can be examined analogously. In the case of a contact drawn out strongly in the y direction in each section $y = \text{const}$ in which neighbourhoods the curvature of the entrance boundary Γ_i is not large, the problem can be reduced with small error to a plane hydrodynamic contact problem. The plane problem obtained in this manner can be investigated by regular pertubation methods /9/.

Plane problems that occur in the sections $y = \text{const}$ were considered above, in with either direct contact between the bodies, or contact between the bodies through an oil interlayer occurs. Using the results obtained, the problem of mixed friction conditions can be considered. Let $a_0(y)$ and $a_p(y)$ be, respectively, the parts of the domain boundaries of the dry and lubricated contacts of rough elastic bodies. Then the bodies making contact will be separated by a lubricant layer in the sections $y = \text{const}$ in which $a_p(y) < a_0(y)$, while we have $a_p(y) = a_0(y)$ in the remaining sections $y = \text{const}$, and the bodies will be directly in contact.

We present a simple example illustrating the geometry of the contact domain elucidated above. We assume the bodies making contact to be smooth and the liquid viscosity to be constant. We consider sections of the contact domain close to the section $y = 0$ in which the change in the function $p_{00}(y)$ can be neglected with small error. Then for oil deficiency conditions in the contact domain sections under consideration, the gap profile will have the form /8/

$$H_0(y) = A_0(V\varepsilon_q^2)^{1/3}|\alpha_1(y)|^{2/3},\quad \alpha_1(y) = (a_p(y) - a_0(y))/\varepsilon_q \tag{2.17}$$

where A_0 is a constant, and $\alpha_1(y)$ characterizes the local remoteness of the entrance boundary from the boundary of the Hertz contact ellipse. By specifying different functions $\alpha_1(y)$ (for instance $\alpha_1(y) = z_*\theta(-z_*)$, $z_* = \cos(10y) - 0.9$, where $\theta(z_*)$ is the Heaviside function), we obtain different lubrication conditions in the contact domain.

The dependence of the lubricant layer thickness in the contact domain on the configuration of the entrance boundary becomes evident from (2.17). It should be noted that analogous behaviour of the lubricant in the contact domain was noted in experiment /10/ for oil deficiency conditions.

Note that as the force P grows, the degree of body roughness k decreases (for fixed a_H, b_H and $a_p(y) - a_0^\circ(y)$, where $a_0^\circ(y)$ is the abscissa of the Hertz contact ellipse boundary for smooth bodies), the parameter λ characterizing the contribution of roughness to the solution of the problem decreases, and $|a_0(y) - a_0^\circ(y)|$ also decreases $(a_0(y) - a_0^\circ(y) < 0)$. Hence, the rough contact zones in which dry contact occurred earlier, will be separated by a lubricant layer as P increases or k decreases. An analogous lubricant behaviour pattern occurs in the contact as $|a_p(y)|$ *increases* $(a_p(y) < 0)$ *for fixed values of* $a_0(y)$ *and* $\lambda \geqslant 0$.

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